

## Stress Analysis of Fiber-reinforced Composite Built-up Beam Using the Unit Cell Method

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### ABSTRACT

Macro- and microscopic levels of analyses are used to capture the behavior of a built-in beam made of a unidirectional fiber-reinforced polymeric material. Finite element formulation based on the displacement approach and minimum potential energy principle is carried out in the longitudinal direction of the beam to determine the lateral displacements (deflections) at various sections, stresses and strains. A unit cell method in micromechanical analysis based on the finite element method is used to define the state and distribution of the stresses induced in transverse sections. Hexagonal fiber-matrix packing system is idealized for the problem. The package of ANSYS 5.4 and a Matlab v6.5 program are used to solve for the macro- and micromechanical analyses. To examine the validity of the presented method, it is applied to analyze a beam made of matrix and fibers with the same elastic properties. The results are then compared with those obtained using the method followed for isotropic materials considering the same values of the elastic properties, the convergence obtained verifies the validity of the analysis method adopted by the current work.

### الخلاصة:

تم في هذا البحث استخدام التحليلات الميكانيكية العيانية والمجهريّة لمتابعة سلوك عتبة مصنوعة من مادة بوليمرية مقواة بألياف أحادية الاتجاه. استخدمت طريقة العناصر المحددة بالاعتماد على مبدأي الإزاحة والطاقة الكامنة الدنيا لتحديد الإزاحات العرضية والانفعالات والاجهادات في مقاطع مختلفة بالاتجاه الطولي والناجئة من انثناء العتبة موضع البحث. استخدمت طريقتي وحدة الخلية والعناصر المحددة لتحديد نوعية وطبيعة توزيع الاجهادات المتكونة في المقاطع العرضية لهذه العتبة. اعتمد نظام الحشو السداسي للألياف والمادة الأساس لهذا التطبيق ونفذ الحل باستعمال برنامجي (Ansys 5.4 & Matlab v6.5) لحساب المتغيرات المختلفة للمسألة. ولاختبار صلاحية الطريقة الحالية تم تحليل عتبة افتراضية مصنوعة من ألياف و مادة أساس بخواص مرونة موحدة وقورنت نتائجها مع نتائج تحليل اعتمد الطرق الخاصة بالمواد موحدة الخواص مع اعتماد ذات الخواص للعتبة الافتراضية وكان التقارب بينهما يشير إلى صحة الطريقة الحالية. هذا أسلوب متبع في الكثير من البحوث العالمية من هذا النوع.

### INTRODUCTION:

The need to understand the composite materials drives the researches in this field into the micro-mechanics of this type of materials [1&2]. Theoretical and experimental methods are usually applied in micromechanical analysis in order to tackle such problems. The theoretical methods can be

either numerical which provides more or less exact solutions or analytical which makes use of comparatively crude models which enable the problem to be treated in a more general manner and to attain finally some mathematical expression which contain all parameters under consideration. Numerous works have been done on the micromechanical study of the composite structures. **Soykasap, Ö.** [3] presented some micromechanical models for the analysis of bending behavior of woven composites based on the classical lamination theory (CLT). **Babu, E. J. et. al.** [4] presented a three-dimensional model based on the generalized method of cells (GMC) principle. **Shuguang Li** [5,6] and **Li & Zhenmin Zou** [7] presented the unit cell method for micro-mechanical analysis of unidirectional fiber reinforced composites subjected to an axial tensile or compressive load.

The present work introduces a methodology for a static analysis of the stresses macro and microscopically induced in a flexurally loaded fiber-reinforced composite beam. The analysis in a microscopic level uses the method of unit cells adapting the finite element formulation as a numerical solution approach for the problem manipulation.

The assumptions used in solution are :- fibers and matrix are assumed isotropic and homogeneous with perfect bonding between them, and the composite material is considered homogeneous on macroscopic level, and the loads are assumed to be applied at the infinity. The analyses are performed by discretization and meshing the beam in the longitudinal direction to determine the field quantities (stresses, strains & displacements), then discretization and meshing in a transverse direction with a suitable unit cell and investigating the effects of fiber cross section, fibers volume fraction and elastic properties on the stresses, strains and displacements of the unit cell and the entire composite.

#### **FINITE ELEMENT FORMULATION FOR THE BEAM:**

A built-in composite beam of length ( $L$ ) and a circular cross-section of diameter ( $d$ ) is to be considered. The beam is transversely loaded by two concentrated forces ( $p/2$ ) equi-spaced from both ends and from each other. Displacement approach of the finite element is adopted to formulate the problem. The beam is discretized into three elements (length,  $l = L/3$ ) with four nodes. Using the principle of minimization of potential energy [8] and inserting the boundary conditions result into the following set of equations:

$$\frac{E_1 I_c}{l^3} \begin{bmatrix} 4l^2 & -6l & 2l & 0 & 0 & 0 \\ -6l & 24 & 0 & -12 & 6l & 0 \\ 2l^2 & 0 & 8l^2 & -6l & 2l^2 & 0 \\ 0 & 12 & -6l & 24 & 0 & 6l \\ 0 & 6l & 2l^2 & 0 & 8l^2 & 2l^2 \\ 0 & 0 & 0 & 6l & 2l^2 & 4l^2 \end{bmatrix} * \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -p/2 \\ 0 \\ -p/2 \\ 0 \\ 0 \end{Bmatrix} \quad (1)$$

Basing on the bending theory [9] and using chain rule of differentiation [10] and the basics of finite elements principles of one-dimensional element of linear type [11], the bending moment (m) and bending stress can be put as:

$$m = \frac{4E_1 I_c}{l^3} \left[ \frac{d^2}{d\xi^2} \{N_I(\xi)\}^T \{u_i\} \right] \quad (2)$$

$$\sigma = \frac{4E_1 y}{l^3} \left[ \frac{d^2}{d\xi^2} \{N_i(\xi)\}^T \{u_i\} \right] \quad (3)$$

### STRESS ANALYSIS THROUGH BEAM CROSS-SECTION:

The hexagonal fiber-matrix packing system is adopted for the idealization of the material of the composite beam under consideration due to its advantages, as it has more symmetry transformations, and it preserves the property of transverse isotropy, and it is characterized by its compactness compared with other packing systems[6]. The hexagonal layout has what so called the periodic element translations of which in y- and z-direction can cover the whole area of the cross-section (Fig.1). The size of this periodic element can be reduced to quarter of it, which can also be further reduced to a unit cell size by virtue of the translational and reflectional symmetry transformations existed in the system.

### THE EFFECTIVE PROPERTIES OF THE UNIT CELL:

A three-dimensional model is designated for the unit cell representing the composite material under consideration. The effective elastic properties of the unit cell are calculated by the rule of mixture and Halpin-Tsai equation [12]:

$$E_2 = \frac{E_f E_m}{E_f V_m + E_m V_f} \quad \text{and} \quad E_1 = E_f V_f + E_m V_m \quad (4)$$

$$v_{12} = v_f V_f + v_m V_m \quad (5)$$

$$G_{12} = \frac{G_f G_m}{G_f V_m + G_m V_f} \quad \text{and} \quad G_{23} = \frac{E_2}{2(1+v_{23})} \quad (6)$$

$$\frac{M}{M_m} = \frac{1+\xi\eta V_f}{1-\eta V_f} \quad \text{where:} \quad \eta = \frac{(M_f / M_m) - 1}{(M_f / M_m) + \xi} \quad (7)$$

The value of  $\xi$  (a measure of fiber reinforcement of the composite which depends on the fiber geometry, packing geometry & loading conditions) can be determined using equation of Hewitt & Malherbe [13] :

$$\xi = 1 + 40 V_f^{10} \quad (8)$$

### **THE AREAS AND FORCES APPLIED ON THE UNIT CELL:**

A unit cell of trapezoidal geometry shown in Fig.2 is chosen at the node possessing the maximum longitudinal (bending) stresses throughout the beam to capture the variation and/or distribution of the field parameters (stress, strain and displacement). Areas of various sides of the unit cells in the directions of x, y and z are mentioned in the fig.2 where (b) can be found using the following relation [6]:

$$V_f = \frac{\pi a^2}{2\sqrt{3} b^2} \quad (9)$$

Then, the forces applied on the various sides due to the stresses can be calculated by multiplying the area by the stress. The unit cell referred to above is discretized and meshed by the 3-dimensional element of the type of 8-node brick element known as solid-45 in the ANSYS program default as shown in Fig.3.

### **MICRO-STRESSES OF THE FIBERS AND MATRIX:**

The micro stresses of the fibers and matrix for the various cases can be determined by using the principle of equating the elastic strains in the composite material with those in the matrix and fibers, this means [14]:

$$\epsilon_c = \epsilon_f = \epsilon_m \quad (10)$$

Then, using Hook's law in 3-dimensional state of stress to calculate the normal microstresses in terms of their corresponding normal elastic strains. Thus for normal stresses of fibers and matrix [15&9]:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} = \begin{bmatrix} \lambda + 2G & \lambda & \lambda \\ \lambda & \lambda + 2G & \lambda \\ \lambda & \lambda & \lambda + 2G \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{Bmatrix} \quad (11)$$

Where:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad (12)$$

$$G = \frac{E}{2(1+\nu)}$$

The micro-stresses of fibers and matrix and for the unit cells are determined at those nodes which represent the midpoints between the fiber and its surrounding matrix and the direct next fibers from all directions (Fig. 4).

## RESULTS AND DISCUSSION:

The presented results are obtained from the finite element formulation (FEF) for a unidirectional fiber-reinforced composite beam (UDFRCB) using a Matlab program and Ansys 5.4 for analyzing the macro- and micro-stresses induced in a built-in beam. The elastic properties of the constituent materials considered (E-glass as a fiber and polyster as a matrix) are tabulated in table (1). The elastic properties of the composite materials, tabulated in table(2), are calculated using the equations mentioned before.

### MACRO-MECHANICAL ANALYSIS:

ANSYS V-5.4 was used to solve for the beam response and stresses. The beam is of circular cross-section of diameter (50 mm) reinforced by unidirectional fibers with different volume fractions and subjected to two bending forces of 3KN. The type of the element by which the cylindrical beam under consideration longitudinally meshed in ANSYS is solid 95 [16] and in MATLAB is the linear one-dimensional [17&18].

### Effect of Variation of Fiber Volume Fraction on Beam Responses and Stresses:

Figures 5 and 6 show the stresses and deflection of the beam under bending forces of 3KN at different fiber volume fractions ( $V_f$ ). The values of 40% and 50% seem to be the optimum values of reinforcement with regard to the stresses induced in beam material. With respect to the deflection, the situation is rather different, such that it is decreased with the increment of fiber volume fraction due to increasing of the total bending stiffness ( $E_1I$ ) of the beam.

### MICROMECHANICAL ANALYSIS:

A micromechanical analysis is carried out by considering a critical cross-section containing the maximum bending stress induced in the beam material and it is found located at the support. The unit cell method and the finite element technique are used to find the stresses, strains and displacement around and at the fibers and the matrix. Through this analysis the effect of the fiber volume fraction and the fiber diameter are investigated.

It is clearly seen from observation of the pertinent figures that the unit cells of fiber volume fraction of 40-50% represent the minimum values of field parameters (this is also referred to in ref. no.12). Fig.7 shows that there is a slack effect of the fiber volume fraction on the macroscopic normal stresses ( $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$ ) when its values range between (10-60%), but the situation is rather different when  $V_f$  is higher. Whatever  $\sigma_z$  has the lowest affect among the three stresses. A reversed situation holds with respect to the macroscopic displacement of the unit cell, such that, they are decreasing with the increase of fiber volume fraction (Figs.8 & 9). Figs.10 and 11 show the effect of fiber volume fraction on micro stresses of matrix and fibers respectively.

The effect of fiber diameter at constant fiber volume fraction on the maximum unit cell normal stresses is displayed in Fig.12. The overall indication that can be drawn is the inverse proportionality of these stresses with the increase of fiber diameter.

**VERIFICATION OF THE UNIT CELL ANALYSIS RESULTS:**

To verify the validity of the unit cell results, the fibers and matrix are given the same linearly elastic and isotropic properties, one should expect the unit cell to behave as a homogeneous, isotropic and linearly elastic material, e.g. a uniformly uniaxial stress should result in uniform strains in the unit cell related to the applied stress by Hook's law [7]. The results of such a unit cell should also be reasonably close to their corresponding in the orthotropic counterpart [6]. Accordingly, a unit cell of 40% fiber volume fraction was chosen and given isotropic properties and subjected to the same state of stress as that of the orthotropic one. The results obtained are as listed in table (3) along with those of the orthotropic case for the purpose of comparison. It is clearly seen that both results are reasonably close to each other, so the analysis is valid and can be accepted. The values of the resulted stresses which demonstrated in table (3) seem to be very high as the verification analysis was based on the elastic properties of fibers.

Table (1): The elastic properties of the constituent materials: E-glass as a fiber and polyester as a matrix [9]:

$E_f$ (GPa)	$\nu_f$	$G_f$ (GPa)	$E_m$ (GPa)	$G_m$ (GPa)	$\nu_m$
72	0.22	30	3.2	1.175	0.36

Table (2): Elastic properties of E-Glass/Polyester at different fiber volume fractions

$V_f$ %	$E_1$ Gpa	$E_2$ Gpa	$\nu_{12}$	$G_{12}$ Gpa	$\nu_{23}$	$G_{23}$ Gpa
90	65.12	22.86	0.234	8.687	0.345	8.5
60	44.48	7.5	0.35	2.77	0.36	3
50	37.6	6.13	0.29	2.29	0.28	2.4
40	30.72	5.2	0.304	1.908	0.29	2.016
30	23.84	4.48	0.318	1.65	0.311	1.7
10	10.1	3.54	0.346	1.29	0.361	1.31

Table (3): Results of isotropic and orthotropic unit cells

	$U_x$ mm	$U_y$ mm	$U_z$ mm	$\sigma_x$ GPa	$\sigma_y$ GPa	$\sigma_z$ GPa
Isotropic case	0.007	0.0103	0.004	27	40	17
Orthotropic case	0.01	0.02	0.02	31	44	22

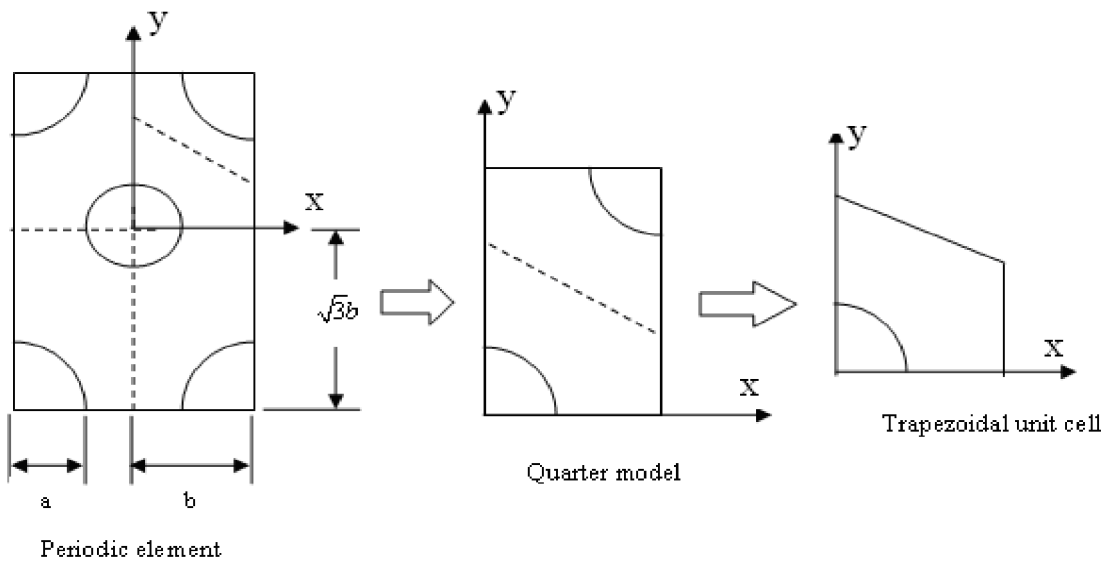


Figure (1): Periodical element, quarter model and the unit cell obtained from the hexagonal system

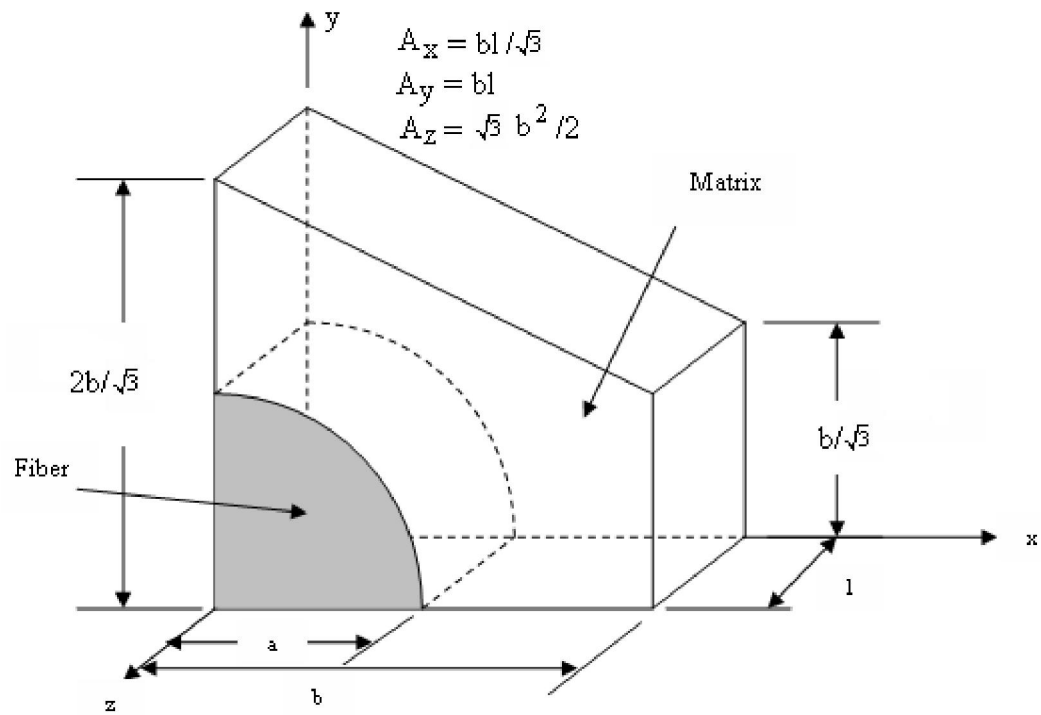
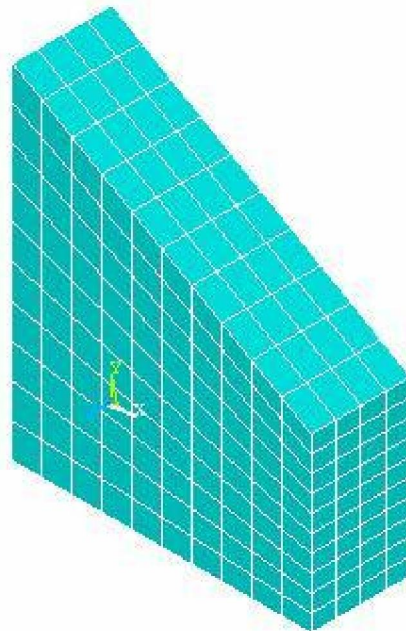


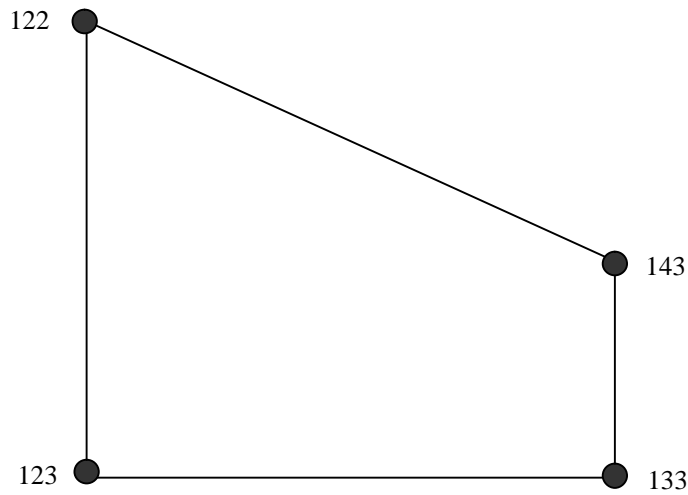
Fig.2 : The three-dimensional unit cell in a quarter model.



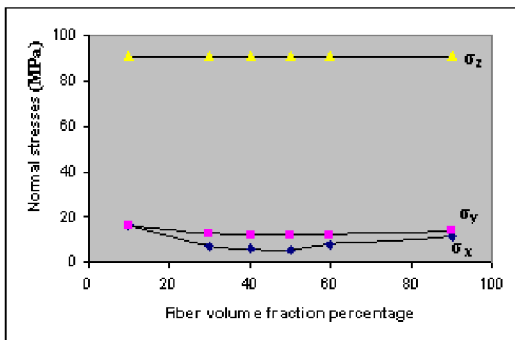
ANSYS

Fig.3: Descritizationof the trapezoidal unit cell

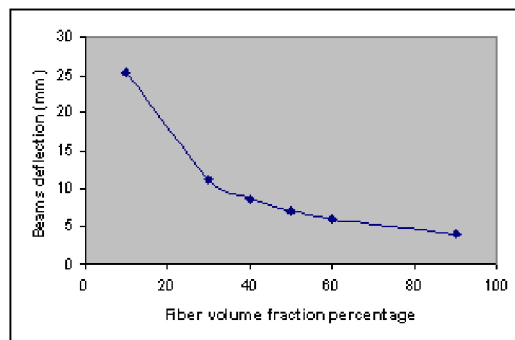




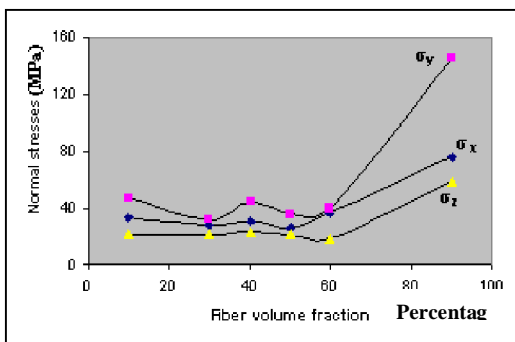
**Fig. 4:** Specification of the corner nodes of the unit cell at which the elastic strains are taken for microstresses calculation.



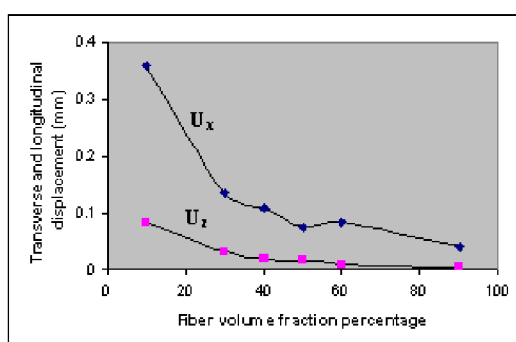
**Fig.(5):** Variation of normal stresses with fiber volume fraction



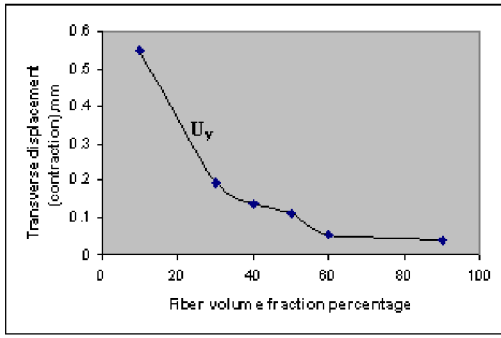
**Fig.(6):** Variation of beams deflection with fiber volume fraction



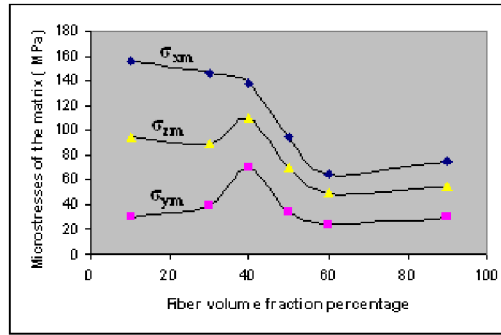
**Fig.(7):** Variation of unit cell normal stresses with fiber volume fraction (at fiber diameter of 0.25mm)



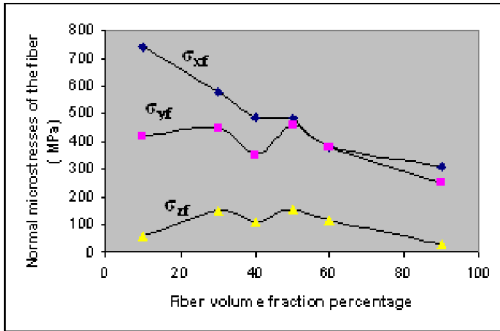
**Fig.(8)** Variation of transverse and longitudinal displacement of the unit cell with fiber volume fraction (at fiber diameter of 0.25mm)



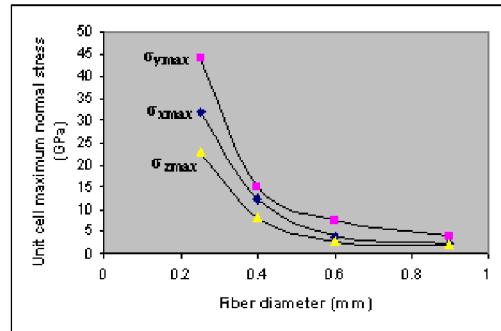
**Fig. (9):** Variation of transverse displacement with fiber volume fraction



**Fig. (10):** Variation of the matrix normal stresses with fiber volume fraction.



**Fig. (11):** Variation of the fiber normal stresses with fiber volume fraction



**Fig. (12):** Variation of unit cell normal stresses with fiber diameter ( at fiber volume fraction of 40%)

## NOMENCLATURE:

Symbol	Meaning	Unit
a	Fiber radius.	m
$A_x, A_y, A_z$	Area of a unit cell face in x,y,z directions	$m^2$
b	Half of center-to-center distance between two adjacent fibers.	m
$E_f, E_m$	Modulus of elasticity of a fiber and of a matrix.	$N/m^2$
$E_1, E_2, E_3$	Modulus of elasticity in material principal directions	$N/m^2$
$G_f, G_m$	Shear modulus of a fiber and of a matrix.	$N/m^2$
$G_{12}, G_{23}, G_{13}$	Shear modulus of a composite in 1-2, 2-3, 1-3 planes respectively.	$N/m^2$
I	Second moment of a cross-sectional area of a beam.	$m^4$
m	Bending moment applied on the beam	N.m
M	Composite modulus $E_2, G_{23}, V_{23}$	
$M_f$	Corresponding fiber modulus $E_f, G_f, V_f$ .	
$M_m$	Corresponding matrix modulus $E_m, G_m, V_m$ .	
N	Shape function	---
u, v, w	Displacements in x, y, and z-directions	m

$V_f, V_m$	Fiber and matrix volume fractions.	---
X, Y & Z	Cartesian coordinate axes	---
$x_1, x_2 \& x_3$	Local coordinate axes.	---
$\epsilon_c$	Normal strain in the composite material	---
$\epsilon_f, \epsilon_m$	Normal strain in the fiber and in the matrix	---
$\epsilon_x, \epsilon_y$	Normal strain in x- and y-directions respectively	
$\eta$	Reduced factor	---
$\xi$	Reinforcement or fiber geometry	---
$\nu_f, \nu_m$	Poisson's ratio of a fiber and of a matrix	---
$\nu_{12}$	Poisson's ratio of a composite for transverse strain in 2-direction and longitudinal strain in 1-direction	---
$\sigma_c, \sigma_f, \sigma_m$	Normal stress in the composite material, fiber, matrix	N/m <sup>2</sup>

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